Lecture 28

Thursday, December 10, 2020 4:56 PM

$$x' = Px$$

$$P \longrightarrow \boxed{\lambda} = a + ib \longrightarrow \boxed{v = v_1 + iv_2}$$

$$v = \begin{bmatrix} 1-2i \\ 3+4i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{v_1} + i \underbrace{\begin{bmatrix} -2 \\ 4 \end{bmatrix}}_{v_2}$$

$$x = \underline{e^{\lambda t}} v = e^{\underline{(a+ib)}t} (\underline{v_1 + iv_2})$$

$$= e^{at+ibt} (v_1 + iv_2)$$

$$= e^{at} (\cos bt + i \sin bt)(v_1 + iv_2)$$

$$= e^{at} \left[(\cos bt \; v_1 - \sin bt \; v_2) + i (\sin bt \; v_1 + \cos bt \; v_2) \right]$$

$$e^{\lambda t} v = x(t) = \boxed{e^{at}(\cos bt \; v_1 - \sin bt \; v_2)} + i \boxed{e^{at}(\sin bt \; v_1 + \cos bt \; v_2)}$$

P real matrix

$$\lambda = a + ib \longrightarrow v = v_1 + iv_2$$

$$\underline{\bar{\lambda}} = a - ib \longrightarrow \underline{\bar{v}} = v_1 - iv_2$$

$$\bar{x}(t) = e^{\bar{\lambda} t} \bar{v} = e^{at}(\cos bt \; v_1 - \sin bt \; v_2) - i e^{at}(\sin bt \; v_1 + \cos bt \; v_2)$$

$$x(t) = \underline{e^{at}(\cos bt\, v_1 - \sin bt\, v_2)} + i\, e^{at}\underline{(\sin bt\, v_1 + \cos bt\, v_2)} \leftarrow$$

$$- \qquad \bar{x}(t) = e^{at}(\cos bt\, v_1 - \sin bt\, v_2) - i\, e^{at}(\sin bt\, v_1 + \cos bt\, v_2) \leftarrow$$

$$\frac{x+\bar{x}}{2} = x^{(1)} = e^{at}(\cos bt\, v_1 - \sin bt\, v_2)$$

$$\frac{x-\bar{x}}{2i} = x^{(2)} = e^{at}(\sin bt\, v_1 + \cos bt\, v_2)$$

$$x' = Px$$

$$x' = ax \rightsquigarrow x = ce^{at}$$

$$\underset{\uparrow}{x'} = Px \longrightarrow \underset{\substack{\uparrow \\ n\times 1}}{x} = \underbrace{e^{Pt}}_{n\times n}\ \underbrace{c}_{n\times 1}$$

$$x(0) = \underbrace{\begin{bmatrix} \vdots \end{bmatrix}}_{n\times 1}$$

How to exponentiate a matrix?

A $n\times n$, what is e^{A}?

$$e^2 = \underset{\frown}{e}(\underset{\smile}{e})$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\Big\downarrow$$

$$e^A \overset{\text{def}}{=} I + \underbrace{\frac{A}{1!}} + \underbrace{\frac{A^2}{2!}} + \frac{A^3}{3!} + \cdots$$

$$e^{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{1!}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 + \frac{1}{3!}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^3 + \cdots \longleftarrow$$

Special case

$$\underbrace{e^{A} = \begin{bmatrix} e^{\lambda_1} & & & 0 \\ & e^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n} \end{bmatrix}} \qquad A^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}$$

$$\longrightarrow e^A = I_n + \frac{1}{1!}A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

$$= \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} + \frac{1}{1!}\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} \lambda_1^3 & & & \\ & \lambda_2^3 & & \\ & & \ddots & \\ & & & \lambda_n^3 \end{bmatrix} + \cdots$$

$$e^A = \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix} \longleftarrow 1 + \frac{1}{1!}\lambda_1 + \frac{1}{2!}\lambda_1^2 + \cdots = e^{\lambda_1}$$

What if A is not diagonal?

* If A is diagonalizable then $A = \underset{\uparrow \atop \text{invertible}}{P}\,\underset{\nwarrow \atop \text{diag.}}{D}\,P^{-1}$.

$$A^2 = AA = PD\underbrace{\cancel{P^{-1}P}}DP^{-1} = PD^2P^{-1}$$

$$A^3 = A^2A = PD^2\underbrace{P^{-1}P}DP^{-1} = PD^3P^{-1}$$

$$\boxed{A^k = PD^kP^{-1}}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = PDP^{-1}$$

$$A^{100}$$

$$D^{100} = \begin{bmatrix} \lambda_1^{100} & \\ & \lambda_2^{100} \end{bmatrix}$$

$$\boxed{e^A = I_n + \frac{1}{1!}A + \frac{1}{2!}A^2 + \cdots}$$

$$= PI_nP^{-1} + \frac{1}{1!}PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + \cdots$$

$$= P\underbrace{\left(I_n + \frac{1}{1!}D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots\right)}_{e^D}P^{-1}$$

$$e^A = Pe^DP^{-1} = P\begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix}P^{-1}.$$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Find e^A.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{cases} x' = Ax \\ x(0) = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \end{cases}$$

$x^{(1)}$ $\quad x^{(2)} = e^{\lambda_2 t} v_2$

$t=0: \; x(0) = \underbrace{e^{A0}}_{I} c$

$x(t) = e^{At} c$

$$\boxed{\begin{bmatrix} -2 \\ -3 \end{bmatrix} = c}$$

$$tA = tPDP^{-1} = P(tD)P^{-1}$$

$$e^{tA} = P e^{tD} P^{-1}$$

$$tD = \begin{bmatrix} t\lambda_1 & 0 \\ 0 & t\lambda_2 \end{bmatrix}$$

$$\boxed{e^{tA} = P \begin{bmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{bmatrix} P^{-1}}$$